FLOW PAST WINGS WITH SIDESLIP

(OBTEKANIE KRYL'EV, LETIASHCHIKH SO SKOL'ZHENIEM)

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M. F. PRITULO
(Moscow)

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A method is given for determining the supplementary loading due to sideslip. The method is based on studies of the nonlinear equation for the velocity potential. The case in which, with $M > 1$, the method of small parameters reduces in second approximation to infinitely large first derivatives is dealt with in greater detail.

When calculating the pressure on a wing under conditions of sideslip at supersonic velocity it is customary to solve the linearized equation for the velocity potential. In order to linearize the equation, it is sufficient to assume only a small angle of incidence. No limits are imposed on the actual angle of sideslip itself. The problem is somewhat more difficult than that of flow past a wing, symmetrical about a longitudinal axis, without sideslip. As a rule, however, the most interesting cases occur when the angle of sideslip is of the same order as the incidence. With small incidence, then, the supplementary pressure caused by the sideslip becomes a quantity of second order of magnitude as compared with the pressure over a wing without sideslip. When determining the supplementary pressure this circumstance allows one to deal with small changes in the shape of the body in the flow. This simplification in boundary conditions is only valid for the case when the solution can be written down explicitly as a function of local angles of incidence. In cases where this is not possible (for instance when dealing with flow past a wing with subsonic leading edges) the assumption of small angles of sideslip does not simplify the problem.

When the angle of incidence and that of sideslip are of the same order of magnitude another way is open. One can try to keep the regions where the boundary conditions are known the same as for flow without sideslip. This is accomplished by choosing suitable body axes. The differential equation, defining the solution to the problem, can change in this case.

1. Equation for the supplementary potential and boundary conditions in coordinate axes fixed to the wing. We neglect third-order terms and regard the flow as potential both at subsonic and supersonic velocities. Let Φ be the velocity potential. We determine the supplementary velocity potential ϕ from the equation

$$\varphi = \frac{1}{U_\infty}(\Phi - Ux) - \alpha y - \beta z$$

(Fig. 1 shows the symbol designations) and arrive at the following equation:

$$(M^2-1)\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} = -M^4(\varkappa+1)\frac{\partial^2\varphi}{\partial x^2}\frac{\partial\varphi}{\partial x} + 2M^2(M^2-1)\frac{\partial^2\varphi}{\partial x^2}\frac{\partial\varphi}{\partial x} -$$

$$-2M^2\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{\partial\varphi}{\partial y} - 2M^2\frac{\partial^2\varphi}{\partial x\partial z}\frac{\partial\varphi}{\partial z} - 2M^2\alpha\frac{\partial^2\varphi}{\partial x\partial y} - 2M^2\beta\frac{\partial^2\varphi}{\partial x\,\partial z} \qquad (1.1)$$

In Equation (1.1) the M number refers to the approaching stream U_∞.

We write down the boundary conditions for the wing to the same order of accuracy. The wing surface is only partly in plane $y = 0$. The condition of gas flow past [through] this surface is of the form

$$\alpha + \partial\varphi/\partial y = 0 \qquad (1.2)$$

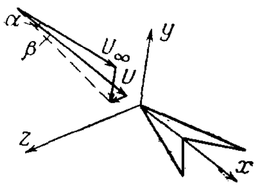

Fig. 1.

It is evident that the boundary conditions on the wing are independent of sideslip angle.

If we apply the small-parameter method to Equation (1), we set $\phi = \phi_1 + \phi_2 + \phi_2' + \ldots$ (the subscript indicates the order of magnitude). Second-order quantities are expressed as the sum of two terms, one of which, ϕ_2, is equal to a second-order quantity in the velocity potential round the wing, without sideslip; functions ϕ_1, ϕ_2 and ϕ_2' satisfy the following equations:

$$(M^2-1)\frac{\partial^2\varphi_1}{\partial x^2} - \frac{\partial^2\varphi_1}{\partial y^2} - \frac{\partial^2\varphi_1}{\partial z^2} = 0 \qquad (1.3)$$

$$(M^2-1)\frac{\partial^2\varphi_2}{\partial x^2} - \frac{\partial^2\varphi_2}{\partial y^2} - \frac{\partial^2\varphi_2}{\partial z^2} = -M^4(\varkappa+1)\frac{\partial^2\varphi_1}{\partial x^2}\frac{\partial\varphi_1}{\partial x} +$$

$$+2M^2(M^2-1)\frac{\partial^2\varphi_1}{\partial x^2}\frac{\partial\varphi_1}{\partial x} - 2M^2\frac{\partial^2\varphi_1}{\partial x\partial y}\frac{\partial\varphi_1}{\partial y} - 2M^2\frac{\partial^2\varphi_1}{\partial x\,\partial z}\frac{\partial\varphi_1}{\partial z} - 2M^2\alpha\frac{\partial^2\varphi_1}{\partial x\partial y} \qquad (1.4)$$

$$(M^2-1)\frac{\partial^2\varphi_2'}{\partial x^2} - \frac{\partial^2\varphi_2'}{\partial y^2} - \frac{\partial^2\varphi_2'}{\partial z^2} = -2M^2\beta\frac{\partial^2\varphi_1}{\partial x\partial z} \qquad (1.5)$$

On the wing surface

$$\frac{\partial \varphi_1}{\partial y} = -\alpha, \qquad \frac{\partial \varphi_2}{\partial y} = 0, \qquad \frac{\partial \varphi_2'}{\partial y} = 0 \tag{1.6}$$

The formula for calculating the pressure will change with the differential equation for supplementary pressure ϕ in the wing system of coordinates. To a second order, inclusive

$$p^0 = \frac{p - p_\infty}{{}^1/_2\, \rho_\infty U_\infty^{\,2}} = -2\frac{\partial \varphi}{\partial x} + (M^2 - 1)\left(\frac{\partial \varphi}{\partial x}\right)^2 - \left(\frac{\partial \varphi}{\partial y}\right)^2 - \left(\frac{\partial \varphi}{\partial z}\right)^2 - 2\alpha\frac{\partial \varphi}{\partial y} - 2\beta\frac{\partial \varphi}{\partial z} \tag{1.7}$$

For the first approximation

$$p^\circ = -2\partial\varphi_1 / \partial x \tag{1.8}$$

It follows from (1.3), the first formula (1.6) and from (1.8) that the effect of a small sideslip angle is to bring about changes in velocity and pressure to a second order. The sum $\phi_1 + \phi_2$ is a solution to the problem of flow past a wing without slip. Supplementary pressure caused by sideslip is determined from

$$p_\beta{}^0 = -2\frac{\partial \varphi_2'}{\partial x} - 2\beta\frac{\partial \varphi_1}{\partial z} \tag{1.9}$$

Formula (1.9) demonstrates that on the vortex sheet which corresponds to the wing in a flow without sideslip we have

$$\frac{\partial \varphi_2'}{\partial x} = -\beta\frac{\partial \varphi_1}{\partial z} \tag{1.10}$$

i.e. the derivative $\partial \phi_2'/\partial x$ displays a discontinuity when passing through the vortex sheet.

If the solution to flow past the wing without sideslip is already known, supplementary pressure due to sideslip has to be determined by solving the nonhomogeneous equation (1.5) for the conditions in the third of the formulas (1.6) and (1.10).

2. Determination of potential for $M > 1$ with finite first derivatives. When dealing with supersonic approach stream velocities the problem of finding a solution to Equation (1.5) is made difficult because the term on the right-hand side, in a whole series of cases close to the surface, assumes infinitely large values. On a delta wing with subsonic leading edges, as we get close to the characteristic surface which divides the turbulent flow from the undisturbed flow, the second derivatives of potential ϕ tend to infinity as

$$\frac{1}{\sqrt{x^2 - (M^2 - 1)(y^2 + z^2)}}$$

Because of this, the assumption that $\beta \partial^2 \phi_1 / \partial x \partial z$ is a second-order quantity is doubtful, though it was valid in deriving Equation (1.5). The same should be said of the first derivatives $\phi_2{}''$ of the supplementary velocities arising from sideslip, which in a whole region of the flow, can assume very large values.

Peculiarities of this kind have been studied by Lighthill. He evolved a highly-developed small-parameter method applicable to cases where the normal method is divergent. As in [1] we introduce the new coordinates

$$x_1 = x + \nu_1(x, y, z), \quad y_1 = y + \nu_2(x, y, z), \quad z_1 = z + \nu_3(x, y, z) \quad (2.1)$$

where ν_1, ν_2, ν_3 are small first-order quantities. Below, we will only deal with the first two terms in the small-parameter series which is the solution to Equation (1.1). The aim of the transformation of (2.1) is to choose ν_1, ν_2 and ν_3 such that, in the application of the small-parameter method to the equation for ϕ in the new coordinates, the right-hand side of the equation, determining the second approximation, be finite everywhere.

At the same time as the transformation of (2.1), we introduce a function ψ instead of the potential, using the formula

$$\psi = \varphi + \frac{\partial \varphi}{\partial x} \nu_1 + \frac{\partial \varphi}{\partial y} \nu_2 + \frac{\partial \varphi}{\partial z} \nu_3 \quad (2.2)$$

If we replace the derivatives of ϕ with respect to x, y and z in Equation (1.1) by derivatives of ψ with respect to x_1, y_1 and z_1, and neglect third- and higher-order terms, we obtain, instead of Equation (1.1)

$$(M^2 - 1) \frac{\partial^2 \psi}{\partial x_1{}^2} - \frac{\partial^2 \psi}{\partial y_1{}^2} - \frac{\partial^2 \psi}{\partial z_1{}^2} = - M^4 (\varkappa + 1) \frac{\partial^2 \psi}{\partial x_1{}^2} \frac{\partial \psi}{\partial x_1} +$$
$$+ 2M^2 (M^2 - 1) \frac{\partial^2 \psi}{\partial x_1{}^2} \frac{\partial \psi}{\partial x_1} + 2M^2 \frac{\partial^2 \psi}{\partial x_1 \partial y_1} \frac{\partial \psi}{\partial y_1} -$$
$$- 2M^2 \frac{\partial^2 \psi}{\partial x_1 \partial z_1} \frac{\partial \psi}{\partial z_1} - 2M^2 \alpha \frac{\partial^2 \psi}{\partial x_1 \partial y_1} - 2M^2 \beta \frac{\partial^2 \psi}{\partial x_1 \partial z_1} \quad (2.3)$$

The invariance of Equation (1.1) with respect to transformation (2.1) is evidence of the connection between particular solutions of (1.4) and (1.5) and such transformations of independent variables which, in the new coordinate system, reduce the equations for the potential to linear ones. If

$$\nu_1 \frac{\partial \varphi}{\partial x} + \nu_2 \frac{\partial \varphi}{\partial y} + \nu_3 \frac{\partial \varphi}{\partial z} = -\varphi_2{}' \quad (2.4)$$

then, in Equation (1.1), written in the new variables, the last term disappears. (With regard to $\partial \phi / \partial x$, $\partial \phi / \partial y$ and $\partial \phi / \partial z$ we should only imply first-order quantities which have already been found in solving (1.3).)

The first and second derivatives of ν_1, ν_2 and ν_3 should not have any singularities. The appearance of singularities when differentiating the left-hand side of (2.4) is due to discontinuities in the values of the second derivatives of ϕ_1. Functions ν_1, ν_2 and ν_3 can also be chosen from the condition

$$\nu_1 \frac{\partial \varphi}{\partial x} + \nu_2 \frac{\partial \varphi}{\partial y} + \nu_3 \frac{\partial \varphi}{\partial z} = -(\varphi_2 + \varphi_2')$$

In Lighthill's method the transformations (2.1) are determined from the condition that the right-hand side of the equation for ϕ_2 vanishes only on surface discontinuities of the derivatives of the potential. This leads to an increase in the number of terms on the right-hand side of the equation for ψ_2. The conditions just mentioned allow one to determine transformations which can simplify the problem of finding a solution involving finite derivatives. Additionally, the invariance of (1.1) with respect to transformations (2.1) and (2.2) allows one to give a simple method, emanating from the solution of Equations (1.3), (1.4) and (1.5), for determining the velocity potential, as discussed below,

Assume, in first approximation, the velocities within a certain region to be continuous, while their derivatives are discontinuous. Then, in addition to the conditions (1.6), it is only necessary to satisfy a condition that the potential be continuous. It may turn out that the potential is given just at the surface of discontinuity of second derivatives. This latter case appears, indeed, to be a general one because, in problems which have been dealt with until the present time, surfaces of Equation (1.3) have been characteristic of discontinuous surfaces, so that they can be regarded as boundaries of regions within which solutions are determined. Thus, we will discuss the problem of finding a solution to Equation (1.1) within a given region whose whole boundary or part of it are surfaces of discontinuity of the second derivatives of the first approximation to the accurate solution. At the boundaries of the region only values of the potential are given. Denote by ϕ_1, ϕ_2 and ϕ_2' solutions of Equations (1.3), (1.4) and (1.5) for the given boundary conditions. Furthermore, let ϕ be a solution of the equation of the supplementary potential in the x_1-, y_1-, z_1-coordinate system, so chosen that the first derivatives of the supplementary potential, determined by the small-parameter method, are finite. In the sum of the solutions of Equations (1.3), (1.4) and (1.5) we alter the symbols; instead of x, y, z, we write x_1, y_1, z_1. Then this sum will be the solution of Equation (2.3), i.e.

$$\varphi_1(x_1, y_1, z_1) + \varphi_2(x_1, y_1, z_1) + \varphi_2'(x_1, y_1, z_1) = \psi(x_1, y_1, z_1)$$

According to Formula (2.2) we have

$$\varphi = \psi(x_1\, y_1,\, z_1) - \nu_1 \frac{\partial \varphi_1}{\partial x} - \nu_2 \frac{\partial \varphi_1}{\partial y} - \nu_3 \frac{\partial \varphi_1}{\partial z} \tag{2.5}$$

In first approximation in the system of coordinates defined by (2.1), the equation for ϕ_1 and the boundary conditions have both the same form. This allows us to write down on the right-hand side of Equation (2.5)

$$\frac{\partial \varphi_1}{\partial x},\quad \frac{\partial \varphi_1}{\partial y},\quad \frac{\partial \varphi_1}{\partial z} \text{ instead of } \frac{\partial \varphi}{\partial x},\quad \frac{\partial \varphi}{\partial y},\quad \frac{\partial \varphi}{\partial z}$$

The potential ϕ determined by Equation (2.5) satisfies the equation for the supplementary potential and has finite first derivatives. We will demonstrate that ϕ satisfies the same boundary conditions as $\phi_1 + \phi_2 + \phi_2'$. To this end, in those regions where the derivatives of ϕ_1, ϕ_2 and ϕ_2' are finite, we expand those quantities in a series of powers of ν_1, ν_2 and ν_3. Neglecting third-order and higher terms, we find that

$$\varphi = \varphi_1(x, y, z) + \varphi_2(x, y, z) + \varphi_2'(x, y, z) \tag{2.6}$$

The first derivative on the right-hand side of (2.5) is finite for any x_1, y_1, z_1. Therefore, relation (2.6) will be valid also on the second derivative surface of discontinuity. Thus, ϕ will indeed be the solution of the formulated problem, while Formula (2.5) gives a simple method of finding it, using a solution obtained by the usual small-parameter method. It should be observed that, in the case just considered, on choosing ν_1, ν_2 and ν_3 it is possible to apply the same conditions to these quantities as in [1], i.e. to require that the right-hand side of the second-order potential equation vanishes only at the surface of discontinuity. The transformations (2.1) are determined solely close to these surfaces, and only near to these surfaces does Formula (2.5) give values for the velocity which differ essentially from the first derivatives of the velocity potential obtained by the usual small-parameter method.

3. Examples. *1. Supersonic velocities.* Let us discuss a wing with straight subsonic leading edges (Fig. 2). The trailing edges will be regarded as supersonic, and therefore condition (1.10) will not be invoked. It is easy to see that a solution to Equation (1.5) for condition (1.6) and $\phi_2' = 0$ on the surface $x^2 = (M^2 - 1)(y^2 + z^2)$ will be

$$\varphi_2' = \frac{M^2 \beta}{\tan^2\chi - M^2 + 1}\left(x \frac{\partial \varphi_1}{\partial z} + \tan^2\chi \frac{\partial \varphi_1}{\partial x} z\right) \tag{3.1}$$

It follows from the previous section that the values of ϕ_2' determined by Formula (3.1) can be used directly for calculation of the pressure on the wing. When $y = 0$, $\phi_2' = 0$.

In using solution (3.1) it is easy to find, also, transformation coordinates leading to Equation (1.5) being homogeneous. Evidently, to do this, one should set

$$\nu_1 = \frac{M^2 \beta \operatorname{tg} \chi}{\tan^2 \chi - M^2 + 1} z, \qquad \nu_2 = 0, \qquad \nu_3 = \frac{M^2 \beta}{\tan^2 \chi - M^2 + 1} x$$

The boundary conditions (1.6) and $\phi_2 = 0$ for $x = \sqrt{(m^2 - 1)(y^2 + z^2)}$ in variables x, y, z are of the same form. Therefore, $\phi_2'(x_1, y_1, z_1) = 0$ and $\phi(x_1, y_1, z_1)$ is a solution to the same problem of flow past a wing without sideslip. (Transformations (2.1) do not, in this case, change the sweep-back angle of the leading edges). In going over from coordinates x_1, y_1, z_1 to physical ones there appears a term in the potential, depending on β. Its magnitude coincides with ϕ_2', determined by Formula (3.1).

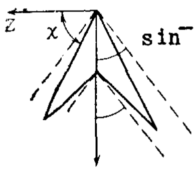

Fig. 2.

2. *Subsonic velocities*. Suppose the leading edge of the wing is part of a straight line coincident with the z-axis. In an incompressible fluid ($M = 0$) ϕ_2' is determined by a homogeneous equation and boundary conditions (1.6) and (1.10). The following is a solution of this equation:

$$\varphi_2' = -\beta \int_0^x \frac{\partial \varphi(\xi, y, z)}{\partial z} d\xi$$

The condition $\partial \phi_2'/\partial x = -\beta \partial \phi_1 / \partial z$ is fulfilled not only on the vortex sheet, but also on the wing. In an incompressible fluid, therefore, supplementary pressure on a wing surface is zero.

In the case of compressible gas we must add the following term to the above expression for ϕ_2':

$$\frac{M^2}{1 - M^2} \left(x \frac{\partial \varphi}{\partial z} - \int_0^x \frac{\partial \varphi}{\partial z} d\xi \right),$$

and this expression is a solution of (1.5) with zero boundary conditions on the wing and on the vortex sheet.

A change in the direction of the vortex sheet by an angle β leads to local change in angles of incidence of an order $a_i \beta$ (a_i is the induced angle of incidence or attack). In flow past high-aspect-ratio wings in an incompressible fluid, neglecting quantities of the order of a_i, β, we can set $\phi_2' \approx 0$. This leads to well-known formulas for calculating the spanwise load distribution on wings under sideslip flight conditions [2]. The particular solution (3.1) allows us to estimate the effect of compressibility of air for wings with straight leading edges to the same order of accuracy.

BIBLIOGRAPHY

1. Lighthill, M.J., The shock strength in supersonic conical fields. *Phil. Mag.* Vol. 40, No. 311, 1949.

2. Queijo, M.J., *Theoretical Span-Load Distributions and Rolling Moments for Sideslipping Wings of Arbitrary Plan Form in Incompressible Flow.* NACA Report No. 1269, 1956.

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